THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

Prof. E. T. WHITTAKER, Sc.D., F.R.S.

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A SIMPLE EXPOSITION OF GRASSMANN'S METHODS.

Expanded from an Address given at the Annual Meeting, January 1927.

BY PROF. R. W. GENESE, M.A.

INTRODUCTION.

IN 1679 Leibniz writing to Huyghens, said: "I am not satisfied with the Algebra (of coordinates), inasmuch as they give neither the shortest ways nor the most beautiful theorems of Geometry; and think that there ought to be a more direct way of dealing with the elements of a figure." He gave a sketch of his ideas, which, however, formed no real contribution.

More than a century later, Möbius took the first step in his *Barycentrische Calcul*, 1827. We are familiar with the formula $\bar{z}=\dfrac{\Sigma mz}{\Sigma m}$ for the distance of a mass-centre from a line or plane; Möbius used the points instead of the distances and wrote

$$G=\frac{\Sigma mA}{\Sigma m}.$$

ILLUSTRATION. P is the point $\dfrac{lA+mB+nC}{l+m+n}$; AP, BP, CP meet BC, CA, AB in D, E, F; FE meets BC in X.

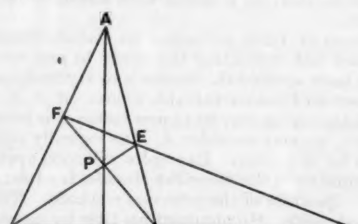


FIG. 1.

From the construction for mass-centre, we must have

$$D=\frac{mB+nC}{m+n},\quad E=\frac{nC+lA}{n+l},\quad F=\frac{lA+mB}{l+m};$$

also $BD:DC=n:m$, etc., showing Ceva's theorem.

Again $$(n+l)E-(l+m)F=nC-mB,$$

i.e. a point on EF = a point on BC.

Hence $$X=\frac{nC-mB}{n-m},$$

$$BX:XC=n:-m,$$

and Menelaus' theorem follows.

Hankel states that we are indebted to Möbius for the use of areal and tetrahedral coordinates.

For the geometry of the line, Möbius laid special stress on the relation $CB=-BC$. Obviously there can never be a calculus applicable to Mechanics without

$$BC+CB=0,$$

whence, also, $$BB=0.$$

Now there are other cases of change of sign on interchange of two *consecutive* letters. Thus, if we obtain a formula for the area of a triangle ABC, we are not surprised if it brings a negative result and we say that

$$ABC=-ACB.$$

And similarly for the tetrahedron.

Grassmann in the *Ausdehnungslehre* (*i.e. Doctrine of Extension*) of 1844, went further, regarding all such magnitudes as *products* and calling them extensives*.

In 1862 he published a revised edition, saying, in the preface, that he had been told that the former book was too philosophical and not sufficiently mathematical. This also made slow progress. The reasons are not hard to find.

1. Grassmann did not take Mathematics at his University (Berlin), where he was a student of Theology (meaning to become a minister) with Literature and Philosophy. He had therefore to make his mark. This he first did by articles of amazing originality on cubic and other curves. (See Whitehead's *Universal Algebra*, page 233.)

2. He dealt at once with extensives of *any* dimensions, using copious new terminology.

3. The drift of the whole was obscure, illustrations being too long postponed.

4. Many are prejudiced against multiplication by a concrete.

5. The geometrical meaning of the sum of two extensives was not given. In the absence of definition, no juggling with letters could afford a proof of the Associative Law.

In 1888 Prof. Peano of Turin published his *Calcolo Geometrico*, founded on Grassmann's methods but restricting the study to real space. Success soon followed. Italians have applied the methods to Hydrodynamics and Electromagnetism. We owe to Peano a valuable axiom—if A, B, magnitudes of the same kind, are such that no matter what new factor X is introduced, we always have $AX=BX$, then, we may consider A, B as logically equivalent, *i.e.* either may be substituted for the other. Examples will soon appear.

In the January number of the *Nouvelles Annales* for 1892, Dr. Carvallo gave a remarkably clear exposition of the principal methods. The article cannot be too strongly recommended. He obtained his title by an essay on the motion of a bicycle, using the methods.

Dr. Whitehead's brilliant *Universal Algebra*, 1897, contains accounts of parts of the *Ausdehnungslehre* and many applications of the methods, with a small bibliography.

* Their immense utility in the development of the Theory of Determinants is shown in the text-book by Scott and Mathews, where, however, they receive the not very intelligible name of "alternate numbers". Chapter XVII contains geometrical applications.

Exposition.

§1.

A, B are fixed points, P any point, $PARB$ a parallelogram.

The fundamental theorem of statics may be expressed, symbolically, thus:

$$PA+PB=PR.$$

Or, since the diagonals bisect each other at M,

$$P(A+B)=2PM.$$

But A, B, M are fixed; hence Peano's axiom applies, and we have (with Möbius)

$$A+B=2M.$$

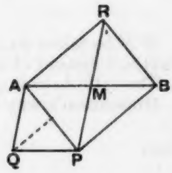


Fig. 2.

Let us test the equivalence, taking another line CD, mid-point N;

$$C+D=2N.$$

Is $$(A+B)(C+D)=4MN$$
a true statement? Now in forces

$$AC+AD=2AN, \quad BC+BD=2BN$$

and $$AN+BN=2MN.$$

Thus the statement is true.

§2.

Again, the moment of a force PA about a point Q is represented by twice the area of the triangle QPA (carrying sign).

The principle of moments now gives us

$$QPA+QPB=2QPM,$$

an obvious geometrical theorem.

The moment QPA vanishes, if Q lies in PA.

Conversely, to prove $QPA=0$ is one of four different ways of showing that Q, P, A are in one line.

Illustration. $ACBD$ is a quadrilateral, EF the third diagonal. The mid-points M, N, L of the diagonals will be in line if

$$\frac{A+B}{2}\cdot\frac{C+D}{2}\cdot\frac{E+F}{2}=0,$$

or, $$(AC+AD+BC+BD)(E+F)=0,$$

i.e. if $$\left.\begin{array}{c} ACE+0+0+BDE \\ +0+ADF+BCF+0 \end{array}\right\}=0.$$

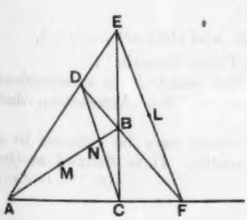


Fig. 3.

Taking ACE as positive, BDE is negative;

$\therefore$ as triangles

$$ACE+BDE=\text{quad}^{l}\ ACBD.$$

And $ADF+BCF=$ same taken negatively;

$\therefore$ sum is zero

and the theorem follows.

Another way is to prove that

$$MN+NL=ML,$$

i.e. a force through $N=ML$,

and the proof turns on $AD+DE=AE$, etc., and may be left to the reader.

Again, let N be any point in AB, and take

PH a fraction l of PA,

PK a fraction m of PB;

and resolve these forces into components along PN and parallel to AB.

Then if $lAN = mNB$, the components parallel to AB cancel, and we obtain

$$lPA + mPB = (l+m)PN$$

and, as before, $$lA + mB = (l+m)N.$$

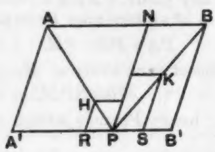


FIG. 4.

If N be taken on AB produced either way, the only change is that we must subtract instead of adding the forces.

The ratio $l : m$ is given (with sign) by $NB : AN$.

Grassmann's way is to assume

$$N = \alpha A + \beta B,$$

then $$AN = 0 + \beta AB,$$

$$NB = \alpha AB + 0;$$

$$\therefore\ AB \,.\, N = AN \,.\, B + NB \,.\, A = AN \,.\, B - BN \,.\, A.$$

Here AB, AN, BN (carrying sign) are treated as scalars, AB being the end of multiplication in a "Two Point Domain", *i.e.* if we are dealing *only* with the *line*, ABC has no meaning and is not required.

The special case of $B - A$, giving a point at infinity, is the most important in the subject, and, like the "couple" in Statics, merits a distinctive name. The name now commonly used is "Vector".

§3.

By introducing a new reference point C, outside AB, we can now deal with any point R in the plane ABC; thus,

$$lPA + mPB + nPC = (l+m)\,PN + nPC = (l+m+n)\,PR,$$

or, $$lA + mB + nC = (l+m)\,N + nC = (l+m+n)\,R,$$

R dividing NC in the ratio $n : l+m$.

Grassmann determines the coefficients for any point R, thus:

$$R = \alpha A + \beta B + \gamma C \text{ say;}$$

$$\therefore\ BCR = \alpha BCA + 0 + 0, \text{ etc.}$$

and since $$BCA = CAB = ABC,$$

we see that α, β, γ are the areal coordinates of R, and that $\alpha + \beta + \gamma = 1$.

ABC is the end of multiplication in a "Three Point Domain".

Since three forces have only one resultant, the point R is independent of the grouping, or order, of the terms, that is, the Associative and Commutative Laws hold for Addition.

In exactly the same way any point in real space may be referred to a tetrahedron $ABCD$, and $ABCD$ with sign is scalar. It is defined as the product of area ABC by distance from it of D.

§4.

Let us now perform a multiplication in a plane:

$$PQ = (\alpha_1 A + \beta_1 B + \gamma_1 C)(\alpha_2 A + \beta_2 B + \gamma_2 C)$$
$$= (\beta_1\gamma_2 - \gamma_1\beta_2)\,BC + (\gamma_1\alpha_2 - \alpha_1\gamma_2)\,CA + (\alpha_1\beta_2 - \beta_1\alpha_2)\,AB,$$

because $AA = 0 = BB = CC$, $CB = -BC$, etc.

The interpretation of this is that a force PQ has been resolved into three components along the sides of ABC.

Doing the same with any number of forces in a plane and adding the components along the same lines,

$$\begin{aligned}\Sigma PQ \text{ takes the form } & lBC + mCA + nAB \\ &= (lB - mA)C + nAB \\ &= (l - m)NC + nAB.\end{aligned}$$

Clearly $lB - mA$ gives a point on the resultant.

Similarly $lC - nA$ gives one, K say: hence the line of action NK of the resultant is determined.

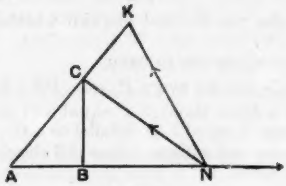


FIG. 5.

In confirmation,

$$\begin{aligned}(lB - mA)(lC - nA) &= l^2BC - lnBA - mlAC + 0 \\ &= l(lBC + mCA + nAB).\end{aligned}$$

The magnitude of the resultant $= \dfrac{(l-m)(l-n)\,NK}{l}$.

If $l = m$, or $l = n$ the system reduces to two parallel forces. And if $l = m = n$, we have a multiple of $BC + CA + AB$, *i.e.* a couple.

§ 5.

In space we shall have

$$\begin{aligned}PQ &= (\alpha_1 A + \beta_1 B + \gamma_1 C + \delta_1 D)(\alpha_2 A + \beta_2 B + \gamma_2 C + \delta_2 D) \\ &= lBC + mCA + nAB + (pA + qB + rC)\,D, \text{ say,}\end{aligned}$$

i.e. a force is resolved into components along the edges of $ABCD$; and ΣPQ takes the same form.

We now have a force RS in the plane ABC together with a force XD say.

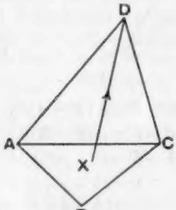


FIG. 6.

These cannot reduce to a single force unless RS passes through X, requiring

$$(lBC + mCA + nAB)(pA + qB + rC) = 0,$$

i.e. $$lp + mq + nr = 0,$$

since $BCA = CAB = ABC$ and other terms vanish.

Two lines of action intersect, or lie in one plane, if

$$(l_1BC + \ldots + p_1AD + \ldots)(l_2BC + \ldots + p_2AD + \ldots) = 0,$$

i.e. if $$(l_1p_2 + p_1l_2) + \ldots + \ldots = 0.$$

The two conditions are introductory to a considerable literature in Line Geometry, Ruled Quadrics, etc.

One easy exercise may be suggested.

Given seven non-intersecting lines in space, they determine *uniquely* the ratios of seven forces which act along the lines and balance.

§ 6.

We now return to the vector, and enquire whether two vectors can be equivalent, $B - A = D - C$.

Applying Peano's axiom, we are to have

$$P(B-A) = P(D-C) \text{ for every } P \quad \text{or} \quad PB - PA = PD - PC.$$

Now the left side is a force through P equal and parallel to AB, and the right side a force through P equal and parallel to CD.

It is therefore necessary and sufficient that AB should be equal and parallel to CD and in the same sense.

Aliter, $$B - A = D - C$$

if $$B + C = A + D.$$

Whence BC, AD have the same mid-points and $ABDC$ is therefore a parallelogram.

Similarly, it may be shown that two vectors (of the form $B - A$) may have *any given finite ratio* provided they are represented on parallel lines.

Conversely, any segments of parallel lines have vectors with a scalar ratio, that of their lengths.

ILLUSTRATION. Maxwell's Reciprocal Diagram.

ABC is any triangle, F the point $\alpha A + \beta B + \gamma C$; PQR is a triangle with its sides parallel to AF, BF, CF; X is the point $\alpha P + \beta Q + \gamma R$. Then XP, XQ, XR are parallel to BC, CA, AB.

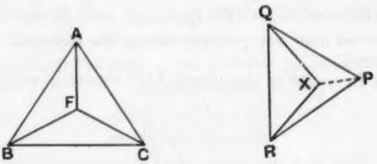


FIG. 7.

We have
$$\left.\begin{aligned} Q - R &= \lambda(F - A) \\ R - P &= \mu(F - B) \\ P - Q &= \nu(F - C) \end{aligned}\right\} \lambda, \mu, \nu \text{ scalars.}$$

Adding,
$$\begin{aligned} 0 &= (\lambda + \mu + \nu)F - \lambda A - \mu B - \nu C \\ &= (\lambda + \mu + \nu)(\alpha A + \beta B + \gamma C) - \lambda A - \mu B - \nu C. \end{aligned}$$

But there can be no relation between A, B, C;

$$\therefore\ (\lambda + \mu + \nu)\alpha = \lambda, \text{ etc.}$$

And now
$$\begin{aligned} X - P &= (\alpha P + \beta Q + \gamma R) - (\alpha + \beta + \gamma)P \\ &= \beta(Q - P) + \gamma(R - P) \\ &= \beta\nu(C - F) + \gamma\mu(F - B) \\ &= \beta\nu(C - B), \text{ because } \beta\nu = \gamma\mu. \end{aligned}$$

Hence the theorem.

§ 7.

We are now able to use a vector u as a factor; thus

$$Au = A(B-A), \text{ say,}$$
$$= AB.$$

Au is sometimes called a "tied vector". The expression of the tied vector as a product forms one of the special merits of Grassmann's system.

It follows that any system of forces in space may be represented by the notation ΣAu.

Consider the special case of two equal unlike parallel forces, viz.: Bv and $-Av$.

Their sum is $(B-A)v$, or, say, uv.

Hence the product of two vectors is a couple.

Aliter, $$(B-A)(C-A) = BC - BA - AC + 0$$
$$= BC + CA + AB,$$

a couple whose moment is twice the area of the triangle ABC, or

$$\bar{u}\,\bar{v}\sin(\widehat{uv}),$$

where $\bar{u}$, $\bar{v}$ are the lengths of the vectors and $(\widehat{uv})$ the angle through which u must be turned *positively* to coincide with v.

Hence, also, $$vu = -uv.$$

Cor. If u, v are parallel, $uv = 0$, and *conversely*.

Illustration. Culmann's construction for the resultant of forces in one plane (Graphic Statics).

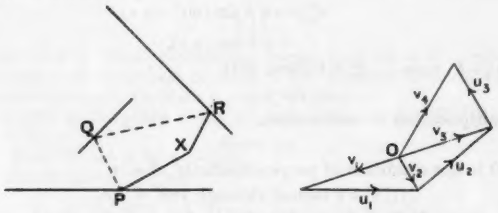


Fig. 8.

It will suffice to take the case of three forces.

Let u_1, u_2, u_3 be the vectors of the forces—placed "end on".

With any convenient origin O, let v_1, v_2, v_3, v_4 be vectors to ends of u_1, u_2, u_3.

If P, Q, R are any points on the lines of action of the forces, then

$$\text{Resultant} = P(v_2 - v_1) + Q(v_3 - v_2) + R(v_4 - v_3)$$
$$= Rv_4 - Pv_1 + (P-Q)v_2 + (Q-R)v_3.$$

If therefore we choose P, Q, R so that

$$P - Q \text{ is parallel to } v_2, \quad Q - R \text{ is parallel to } v_3,$$

forces reduce to $Rv_4 - Pv_1$.

Drawing RX parallel to v_4 and PX to v_1, we obtain a point X on the resultant.

§ 8.

Vectors are added by placing them "end on", for

$$C - A = (C-B) + (B-A),$$

and subtracted by joining ends, for

$$C - B = (C-A) - (B-A).$$

The Distributive Law

$$u(C-A)=u(C-B)+u(B-A)$$

merely states a theorem in projections at right angles to u.

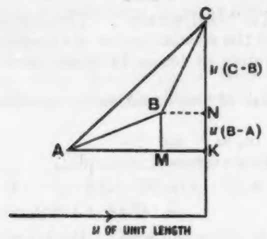


FIG. 9.

In the diagram $MB=KN=u(B-A)$,

$NC=u(C-B)$,

and $KC=u(C-A)$.

§ 9.

If we are dealing only with the plane it is useful (following Peano) to introduce a vector making $+90°$ with v and to denote it by $|\,v$; then

$$u\,|\,v=\bar{u}\,\bar{v}\sin(90°+\widehat{u\,v})$$
$$=\bar{u}\,\bar{v}\cos(\widehat{u\,v}).$$

Since $\cos(-\theta)=\cos(+\theta)$ it follows that

$$v\,|\,u=u\,|\,v$$

and this multiplication is *commutative*.

COR. $v\,|\,v=\bar{v}^2$.

and $u\,|\,v=0$ is the condition of perpendicularity of u, v.

Also $|\,(|\,v)=v$ turned through $180°=-v$.

ILLUSTRATION. If P, Q, R are the centres of squares described externally on the sides of any triangle ABC, then will AP be equal and perpendicular to QR.

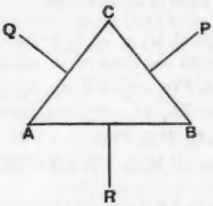


FIG. 10.

If $B-A=u$, $C-A=v$, then

$$Q-A=\tfrac{1}{2}v+|\,\tfrac{1}{2}v,\quad R-A=\tfrac{1}{2}u-|\,\tfrac{1}{2}u;$$
$$\therefore\ Q-R=\tfrac{1}{2}(v-u)+|\,\tfrac{1}{2}(v+u).$$

And

$$P-A=\tfrac{1}{2}(u+v)+|\,\tfrac{1}{2}(u-v);$$
$$\therefore\ Q-R=|\,(P-A).$$

§ 10.

Useful Applications. Origin being fixed, to find formulae for change of oblique axes.

If u, v, u', v' are unit vectors along old and new axes, and x, y, x', y' coordinates of the end of a vector ρ:

$$\rho = xu + yv = x'u' + y'v';$$

$$\therefore\ xuv + 0 = x'u'v + y'v'v,$$

i.e. $$x \sin (\widehat{uv}) = x' \sin (\widehat{u'v}) + y' \sin (\widehat{v'v}).$$

Similarly, $$y \sin (\widehat{uv}) = x' \sin (\widehat{uu'}) + y' \sin (\widehat{uv'}).$$

To find the angle between lines whose equations in trilinears are given.

Lemma. As remarked by the late Dr. Routh (and recently by Mr. R. F. Davis), the equation $l\alpha + m\beta + n\gamma = 0$ states that forces l, m, n along the sides of the triangle of reference have no moment about (α, β, γ), *i.e.* it is the equation to their resultant R.

The vector of R is $lu + mv + nw$, where u, v, w are unit vectors along the sides in order.

Thus $$R^2 = (lu + mv + nw) \mid (lu + mv + nw)$$
$$= l^2 + m^2 + n^2 + 2mn \cos (180° - A) + \text{etc.}$$

Now let ρ_1, ρ_2 be the vectors of two such lines, then

$$\rho_1 \mid \rho_2 = (l_1u + m_1v + n_1w)\,(l_2u + m_2v + n_2w);$$

$$\therefore\ R_1R_2 \cos (\widehat{R_1R_2}) = l_1l_2 + m_1m_2 + n_1n_2 + (m_1n_2 + n_1m_2) \cos (180° - A) + \text{etc.}$$

§ 11.

We come now to what Grassmann terms Regressive Multiplication, viz. when the sum of the numbers of units of two extensives exceeds the number giving a scalar product, *i.e.* 3 for a plane, 4 for space.

The only case in the plane is the product of two lines AB, CD giving their intersection X.

Since X is in CD, let it be $\lambda C + \mu D$.

We are also to have $ABX = 0$;

$$\therefore\ \lambda ABC + \mu ABD = 0, \quad \text{or} \quad \frac{\lambda}{\overline{ABD}} = \frac{\mu}{-\overline{ABC}}.$$

Thus X is given as a mass-point by $\overline{ABD} \,.\, C - \overline{ABC} \,.\, D$.

(The bar $\overline{\quad}$ may conveniently be used to show that a scalar (with sign) is meant; but it may be dropped when there is no doubt about the domain in which one is working.)

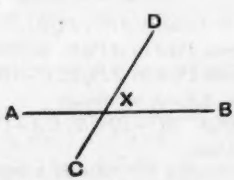


Fig. 11.

The mass is $\overline{ABD} - \overline{ABC}$, *i.e.* twice the area of the quadrilateral $ABCD$, or

$$\overline{AB} \,.\, \overline{CD} \sin (\widehat{AB\,CD}),$$

which is the same as the product of the vectors $B - A$, $D - C$.

Similarly, X can be found in the form $\overline{ACD} \,.\, B - \overline{BCD} \,.\, A$.

COR. $AB \cdot AC = \overline{ABC} \cdot A$ (since $ABA = 0$).

Hence, also, if P be any *point* of the plane

$$P(AB \cdot AC) = \overline{ABC} \cdot PA.$$

Let $$P = lA + mB + nC;$$

then $$PA = mBA + nCA = -mAB - nAC,$$
$$PAC = mBAC = -mABC,$$
$$PAB = nCAB = \quad nABC;$$
$$\therefore\ ABC \cdot PA = PAC \cdot AB - PAB \cdot AC.$$

Writing q, r for AB, AC, we have

$$P(qr) = Pr \cdot q - Pq \cdot r. \quad \text{Cf. § 18.}$$

§ 12.

In space there are two cases:

1. The intersection of a line AB and a plane PQR.

Exactly as in the preceding case it may be shown to be given by

$$APQR \cdot B - BPQR \cdot A.$$

It may also be expressed in terms of P, Q, R by

$$ABQR \cdot P + ABRP \cdot Q + ABPQ \cdot R.$$

For, any point in the plane is given by

$$\overline{PQR} \cdot X = \overline{QRX} \cdot P + \overline{RPX} \cdot Q + \overline{PQX} \cdot R.$$

The triangles are proportional to the tetrahedra $AQRX$, etc., and also to $BQRX$, etc., and therefore to the differences $ABQR$, etc.

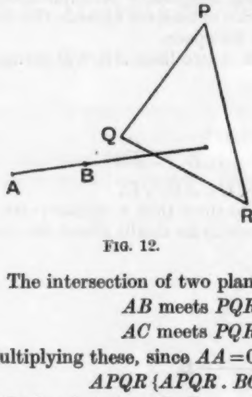


FIG. 12.

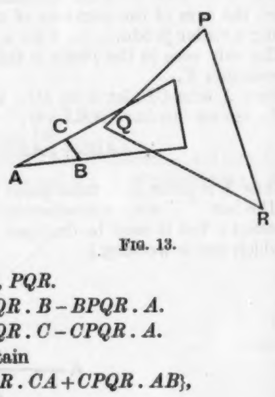


FIG. 13.

2. The intersection of two planes ABC, PQR.

$$AB \text{ meets } PQR \text{ in } APQR \cdot B - BPQR \cdot A.$$
$$AC \text{ meets } PQR \text{ in } APQR \cdot C - CPQR \cdot A.$$

Multiplying these, since $AA = 0$, we obtain

$$APQR\{APQR \cdot BC + BPQR \cdot CA + CPQR \cdot AB\},$$

and the bracket gives the line.

It may be noticed concerning the mass of a regressive product that, as in previous products a new point factor introduced a distance, so in multiplication by a line or plane the sine of an angle springs up, thus:

$AB \times ACD = A$ with mass $\overline{AB} \times \overline{ACD} \times$ sine of their inclination;

$ABC \times ABD =$ unit of AB multiplied by $\overline{ABC}$, $\overline{ABD}$ and sine of their inclination.

Of course the masses of each factor must enter into the result.

Here, as in § 7, a bar is used to imply that a scalar only is intended.

§ 13.

Grassmann's method of dealing with such problems was *masterly*. The course of invention will best be understood if we confine our attention, at first, to the case of the plane determined by three unit points, A, B, C. *Having in mind* the theory of poles and polars, Grassmann imagined that if he could make a line correspond analytically with a point, then the intersection of two lines would correspond to the join of two points. He began by taking BC, CA, AB to correspond with A, B, C respectively, and *assuming that* $BC \,.\, CA \equiv C$, etc., meaning by the sign "$\equiv$" *identity of position*, nothing being implied as to mass. Then he assumed that to $\alpha A + \beta B + \gamma C$ would correspond the line $\alpha BC + \beta CA + \gamma AB$. And now the product of two such lines

$$(\alpha_1 BC + \beta_1 CA + \gamma_1 AB)(\alpha_2 BC + \beta_2 CA + \gamma_2 AB) \equiv (\beta_1\gamma_2 - \gamma_1\beta_2) A + \text{etc.},$$

if the masses above implied are equal.

To this would correspond the line

$$(\beta_1\gamma_2 - \gamma_1\beta_2) BC + \text{etc.},$$

i.e. the line joining the corresponding points. A most encouraging result!

It is interesting to observe (though Grassmann does not seem to have noticed it) that his line *is actually* a polar. For a point P lies on the line $\alpha BC + \text{etc.}$ if

$$\alpha BCP + \beta CAP + \gamma ABP = 0,$$

i.e. if P is a point on the line whose equation in areals is

$$\alpha x + \beta y + \gamma z = 0,$$

which is the polar of the point (α, β, γ) with respect to the conic

$$x^2 + y^2 + z^2 = 0.$$

He now generalises as follows:

Definition. Die Ergänzung, or complement, of a *simple* extensive E (product of units), denoted by $| E$, is the product of all the remaining units taken in such order that

$$E \,|\, E = \text{principal scalar product of the domain}$$
$$= +1, \text{ say.}$$

Thus, for the line AB,

$$| A = B,$$

but $\quad | B = -A$ because $B \,|\, B = -BA = AB$;

for the plane ABC,

$$| A = BC, \; | B = CA, \; | C = AB,$$
$$| BC = A, \text{ etc.};$$

for space referred to the tetrahedron $ABCD$,

$$| A = BCD, \; | B = -ACD, \; | C = ABD, \; | D = -ABC,$$
$$| BC = AD, \text{ etc.}$$

Definition. The complement of a sum is the sum of the complements of its terms.

Assumption. For units of reference only let

$$| A \,.\, | B = | AB, \text{ etc.}, \; | A \,.\, | A = 0.$$

Then $\quad | (\alpha_1 A + \beta_1 B + \ldots) \,.\, | (\alpha_2 A + \beta_2 B + \ldots)$

$$= (\alpha_1\beta_2 - \beta_1\alpha_2) \,|\, A \,|\, B + \ldots$$
$$= | (\alpha_1\beta_2 - \beta_1\alpha_2) AB + \ldots$$
$$= | (\alpha_1 A + \beta_1 B + \ldots)(\alpha_2 A + \beta_2 B + \ldots),$$

or, say $\quad | P \,.\, | Q = | PQ.$

Of course $\quad A \,|\, B = 0,$

because $| B$ must contain A as factor.

If P be a compound unit $=\alpha_1 A+\beta_1 B+\gamma_1 C+$ etc.,

$$P \mid P=\alpha_1^2+\beta_1^2+\gamma_1^2+\dots, \text{ taking } A \mid A=B \mid B=\dots=1.$$

And if $\quad Q=\alpha_2 A+\beta_2 B+\dots,$

then $\quad P \mid Q=\alpha_1\alpha_2+\beta_1\beta_2+\dots$

$\quad =Q \mid P$ by symmetry,

i.e. this kind of multiplication is commutative.

§ 14.

The outstanding application of this theory is to the multiplication of vectors.

1. For the plane.

Vectors being points on the *line* at infinity, the Algebra will be the same as that of the line AB above; but in the interpretation, a length on the line will be replaced, for two unit vectors, by the sine of an angle. Since vectors in the same direction are multiples of each other, it will suffice to examine the relations of unit vectors.

Taking $A-O=i$, $B-O=j$ and at right angles to i, $ij=1$, $\mid i=j$, $\mid j=-i$.

Any other unit vector $\quad P-O=\rho=\cos\theta\,.\,i+\sin\theta\,.\,j,\ \theta=A\hat{O}P.$

Then

$$\mid \rho=\cos\theta\,.\,j+\sin\theta\,.\,(-i)$$
$$=\cos(\tfrac{1}{2}\pi+\theta)\,.\,i+\sin(\theta+\tfrac{1}{2}\pi)\,.\,j,$$

and is therefore unit vector perpendicular to ρ,

$$\rho \mid \rho=\cos^2\theta+\sin^2\theta, \text{ since } ji=-ij$$
$$=\bar{\rho}^2.$$

Taking two vectors,

$$\rho_1=\cos\theta_1\,.\,i+\sin\theta_1\,.\,j,$$
$$\rho_2=\cos\theta_2\,.\,i+\sin\theta_2\,.\,j,$$

we have

$$\rho_1\rho_2=\cos\theta_1\sin\theta_2-\sin\theta_1\cos\theta_2.$$

Assuming the former interpretation of $\rho_1\rho_2$, this is a proof of the formula for $\sin(\theta_2-\theta_1)$.

And,

$$\rho_1 \mid \rho_2=(\cos\theta_1\,.\,i+\sin\theta_1\,.\,j)(\cos\theta_2\,.\,j-\sin\theta_2\,.\,i)$$
$$=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2$$
$$=\cos(\theta_1\sim\theta_2)$$
$$=\rho_2 \mid \rho_1.$$

And the condition of perpendicularity is

$$\rho_1 \mid \rho_2=0.$$

Illustration. A, B, C, D are coplanar points, AB meets CD in E, AC meets BD in F; to show that the circles on AC, BD, EF are coaxal.

Lemma. Any linear relation between points holds also for their vectors from any origin O. Thus,

if $\quad lA+mB+nC=(l+m+n)P,$

then since $\quad lO+mO+nO=(l+m+n)O,$

$$\therefore\ l(A-O)+m(B-O)+n(C-O)=(l+m+n)(P-O).$$

Now let the relation between the points be

$$lA+mB+nC+pD=0,$$

then $\quad lA+mB=-(nC+pD),$

i.e. $\quad$ a point on $AB=$ a point on CD,

either side therefore gives E: similarly we get F.

Taking for origin either point of intersection of the circles on AC, BD, and using small letters for vectors,

$$a \mid c=0,\quad b \mid d=0.$$

Now since $b \mid d=0$,

$$(la+mb) \mid (la+pd)=la \mid (la+mb+pd)$$
$$=la \mid (-nc)$$
$$=0,$$

and the theorem follows.

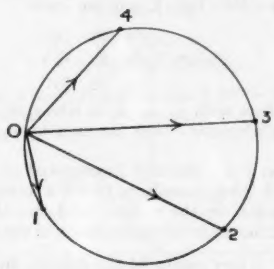
For four points on a line

$$\overline{BC} \,.\, \overline{AD}+\overline{CA} \,.\, \overline{BD}+\overline{AB} \,.\, \overline{CD}=0,$$

therefore for four vectors in a plane

$$\overline{\rho_2\rho_3} \cdot \overline{\rho_1\rho_4}+\overline{\rho_3\rho_1} \cdot \overline{\rho_2\rho_4}+\overline{\rho_1\rho_2} \cdot \overline{\rho_3\rho_4}=0.$$

If the ends of the vectors lie on a circle through the origin O, sin $\widehat{\rho_2\rho_3}$ is proportional to the chord joining the ends of ρ_2, ρ_3, etc.



$\rho_3\rho_1$ negative.

FIG. 14.

The lengths of ρ_1, ρ_2, ρ_3, ρ_4 dividing out, we obtain Ptolemy's theorem (Euc. vi. D).

§ 15.

Vectors in space are points on the plane at infinity, and their Algebra resembles that of points in a plane.* They are (but only for the development of the theory) referred to three mutually perpendicular units i, j, k,

$$\mid i=jk, \mid j=ki, \mid k=ij, ijk=1.$$

Any vector
$$\rho=xi+yj+zk,$$
$$\mid \rho=xjk+yki+zij$$
$$=\frac{1}{x}(xj-yi)(xk-zi).$$

Now $xj-yi=\mid(xi+yj)$ *in plane of* ij, and is therefore perpendicular to $xi+yj$; and, of course, to k; therefore to their plane *which contains* ρ.

Similarly $xk-zi$ is perpendicular to a plane containing ρ.

Hence $\mid \rho$ is a couple (for Pure Mathematics called a bivector) in a plane perpendicular to ρ and its magnitude $=\sqrt{(x^2+y^2+z^2)}=\bar{\rho}$.

Taking two vectors,

$$\rho_1\rho_2=(x_1i+y_1j+z_1k)(x_2i+y_2j+z_2k)$$
$$=(y_1z_2-z_1y_2)jk+\dots.$$

* It is important to observe that the vector-system is independent of the point-system. In the latter, the complement of the vector $B-C$ is $CA-AB=-A(C+B)$, a median *line* and not a vector.

This resolves the couple into three components, and the magnitude of the resultant couple

$$=\sqrt{\{(y_1z_2-z_1y_2)^2+\dots\}}$$
$$=\bar{\rho}_1\bar{\rho}_2\sin(\widehat{\rho_1\rho_2}).$$

And the axis is perpendicular to the plane containing $\rho_1\rho_2$.

Again $$\rho_1 \mid \rho_2=(x_1i+y_1j+z_1k)(x_2jk+y_2ki+z_2ij)$$
$$=x_1x_2+y_1y_2+z_1z_2$$
$$=\bar{\rho}_1\bar{\rho}_2\cos(\widehat{\rho_1\rho_2})$$
$$=\rho_2 \mid \rho_1.$$

Thus if the vectors are at right angles $\rho_1 \mid \rho_2=0$, and conversely.

Cor. $$\rho \mid \rho=\bar{\rho}^2.$$

For three vectors, $ijk=jki=kij=1$, and we have

$$\rho_1\rho_2\rho_3=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

i.e. 6 times the tetrahedron with ρ_1, ρ_2, ρ_3 as edges, or, as Grassmann *assumes*, the volume of the parallelepiped swept through by $\rho_1\rho_2$ if its plane receive a translation ρ_3.

The three-vector theory is naturally interpreted on unit sphere, centre O, and it is convenient, following Hamilton, to use a letter A with two meanings, for the Geometry, a point on the sphere, and, for the Algebra, the vector $A-O$; BC will mean either an arc of great circle, or the bivector $(B-O)(C-O)$.

Illustrations. OP is any radius of the sphere, and the arcs AP, BP, CP meet BC, CA, AB in L, M, N, and, in vectors

$$P=\alpha A+\beta B+\gamma C.$$

Then $$L\equiv\beta B+\gamma C,$$

and $$\beta:\gamma=\sin LC:\sin BL,$$

and similarly for M, N.

The analogue of Ceva's theorem is apparent.

And $$(\gamma C+\alpha A)-(\alpha A+\beta B)=\text{mult. of } M-\text{mult. of } N;$$

$$\therefore\ \gamma C-\beta B \text{ gives the intersection of } BC \text{ and } MN.$$

Whence the analogue of Menelaus' theorem.

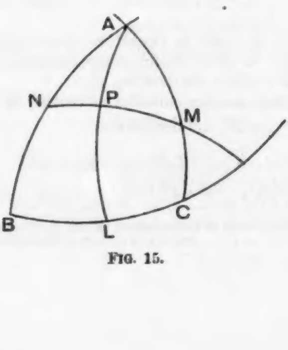


Fig. 15.

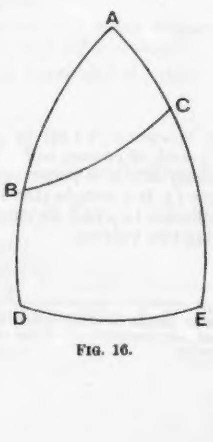


Fig. 16.

Again, let the arcs AB, AC meet the polar circle of A in D, E, so that the arc DE measures the angle A. We have, since $\sin AD = 1 = \sin AE$,

$$B = \cos c \,.\, A + \sin c \,.\, D,$$

$$C = \cos b \,.\, A + \sin b \,.\, E,$$

and therefore, since $A \mid E = 0$ and $D \mid A = 0$,

$$B \mid C = \cos b \cos c + \sin b \sin c \,.\, D \mid E,$$

i.e.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

§ 16.

The interpretation of $\mid u$ depends on the system of axes preferred by the reader. For the system in the diagram

$$u = li + mj + nk,$$

$$\mid u = ljk + mki + nij$$

$$\equiv (lj - mi)(lk - ni),$$

and we see that if jk has a positive aspect *seen from the end of* i, then $\mid u$ appears positive from the end of u.

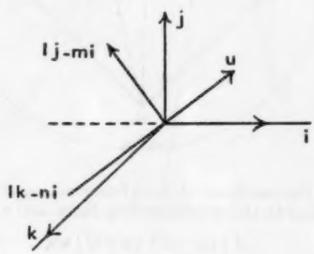


FIG. 17.

If $A'B'C'$ is the polar triangle of the spherical triangle ABC, we have

$$\mid A = \text{unit bivector in plane } B'C',$$

$$\mid BC = A' \text{ with the coefficient } \sin a,$$

$$A \mid BC = \sin a \,.\, AA'.$$

It was proved at the end of § 11, for the plane, that

$$P \,.\, qr = Pr \,.\, q - Pq \,.\, r.$$

Now a line is the complement of a point; and $\mid Q \,.\, \mid R = \mid QR$.

Hence $$P \mid QR = P \mid R \,.\, \mid Q - P \mid Q \,.\, \mid R,$$

and therefore for *vectors*

$$A \mid BC = A \mid C \,.\, \mid B - A \mid B \,.\, \mid C.$$

Adding two similar equations and observing that $A \mid C = C \mid A$, etc.,

$$A \mid BC + B \mid CA + C \mid AB = 0.$$

This linear relation between AA', BB', CC' proves that the perpendiculars of a spherical triangle cointersect.

Also, by introducing lengths, we have for any three vectors

$$u \mid vw + v \mid wu + w \mid uv = 0.$$

The following is a simple application.

$ABCD$ is a tetrahedron, $A-D$, $B-D$, $C-D$ the vectors u, v, w. If BQ is perpendicular to the face ACD, then uw has a positive aspect from a point on QB produced; hence $\mid wu$ will be in the direction BQ.

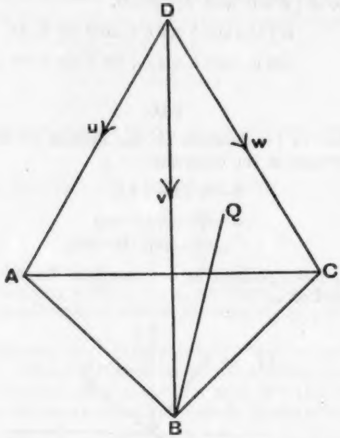


FIG. 18.

We may now find the resultant of three forces acting along the perpendiculars of $ABCD$ proportional to the corresponding faces and all inwards, viz.

$$A \mid vw + B \mid wu + C \mid uv,$$

or, $\quad (D+u) \mid vw + (D+v) \mid wu + (D+w) \mid uv$

$$= D\,(\mid vw + \mid wu + \mid uv), \text{ the other terms cancelling,}$$

$$= D \mid (v-u)\,(w-u)$$

$$= D \mid (B-A)\,(C-A),$$

i.e. a force along the perpendicular from D on ABC proportional to ABC and acting outwards.

§ 17.

EQUATIONS TO LOCI. P being a variable point, the equation

$$(PB \,.\, CA)\,(PC \,.\, AB)\,F = 0$$

expresses that the points Q, R given by the brackets are in line with F; and Grassmann asserts that, because P occurs twice, the locus is of the second degree. This may be verified as follows:

if $\quad P = \alpha A + \beta B + \gamma C,\ ABC = 1,$

then $\quad PB \,.\, CA = \alpha A + \gamma C,$

$$PC \,.\, AB = \alpha A + \beta B,$$

hence $\quad (\alpha\beta AB + \gamma\alpha CA + \gamma\beta CB)\,F = 0,$

or, if fgh are the areal coordinates of F,

$$\alpha\beta h+\gamma\alpha g+\beta\gamma(-f)=0,$$

giving, by properly choosing f, g, h, *any* circum-conic of ABC.

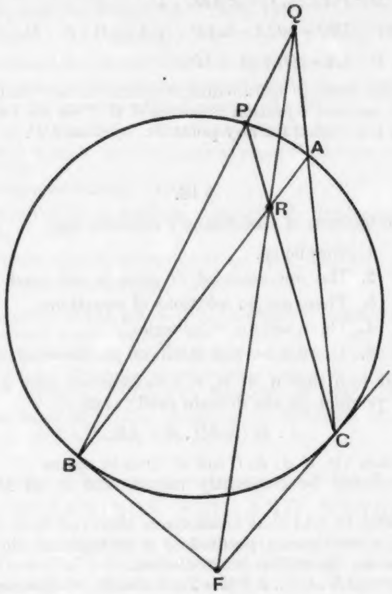


FIG. 19.

It may be seen that F is the pole of BC for the conic, and that the collinearity asserted by the first equation is a case of Pascal's theorem for the hexagon $PBBACC$.

Again, the equation

$$(PA\,.\,l)\,(PB\,.\,m)\,(PC\,.\,n)=0,$$

where l, m, n are lines in the plane ABC and P belongs to this plane, or,

$$(Pl\,.\,A-Al\,.\,P)\,(Pm\,.\,B-Bm\,.\,P)\,(Pn\,.\,C-Cn\,.\,P)=0,$$

or, $Pl\,.\,Pm\,.\,Pn\,.\,ABC-Pl\,.\,Pm\,.\,Cn\,.\,ABP-$ two more terms $=0$,

represents a cubic locus. Grassmann showed that by properly choosing A, B, C, l, m, n this locus might be identified with any cubic whose equation was given in Cartesians—an astonishing performance! An excellent account of this and other cubics may be found in Whitehead's *Universal Algebra*, pp. 233 to 246.

The following is an example in three dimensions.

If l, m, n are three non-intersecting lines in space, the equation

$$PlmnP=0$$

represents the conicoid containing the lines l, m, n.

Replacing l, m by AB, CD and assuming the Associative Law, the equation asserts that if the plane PAB cuts CD at X, then the plane Xn contains the

point P. Hence PX intersects n, and also l, m. Thus P is any point on a transversal of l, m, n; the theorem follows.

Or, we may proceed thus:

$$PAB \,.\, CD = PABD \,.\, C - PABC \,.\, D.$$

Putting $$EF = lBC + mCA + nAB + (pA + qB + rC)\,D,$$

$$P = \alpha A + \beta B + \gamma C + \delta D,$$

the equation to the locus in tetrahedral coordinates may be found.

Obviously the original equation is satisfied if P be on l or n, and it may be shown that it is satisfied for any point $\lambda C + \mu D$ on CD.

§ 18.

The distinctive features of Grassmann's methods are:

1. Simplicity.
2. The operation of division is not used.
3. There are no solutions of equations.
4. There are no imaginaries.
5. There is no real strain on the memory.

The reader will find that if A, B, C be extensives (not necessarily points) and AC a scalar product (in the domain used), then

$$A \,.\, BC = AC \,.\, B - AB \,.\, C$$

in all cases but one, viz. if A, B, C are all lines in space.

The formula should be frequently tested, and is all that needs to be remembered.

It may be useful to add that Grassmann observed that if the number of point factors in a continuous product is a multiple of the number of the domain, then the multiplication is associative.

Thus, in a plane, $AB \,.\, CD \,.\, EF$ $(2+2+2=\text{mult. of } 3)$ is associative, *i.e.*

$$(AB \,.\, CD) \,.\, EF = AB \,.\, (CD \,.\, EF),$$

or $$(ABD \,.\, C - ABC \,.\, D) \,.\, EF = AB \,.\, (CEF \,.\, D - DEF \,.\, C)$$

an identity.

Replacing CD, EF by l, m,

$$AB \,.\, lm = ABl \,.\, m = (Al \,.\, B - Bl \,.\, A)\, m$$

$$= \begin{vmatrix} Al, & Am \\ Bl, & Bm \end{vmatrix}.$$

There is a similar formula for space:

$$ABC \,.\, \alpha\beta\gamma = \{A\alpha, B\beta, C\gamma\}$$

where α, β, γ are planes.

§ 19.

Short Bibliography.

The collected edition, Hermann Grassmanns gesammelte mathematische und physikalische Werke, edited by F. Engel, is in six parts (three volumes, each in two parts, paged separately), issued at intervals from 1894 to 1911. The two parts of the first volume are reprints of the two editions of the *Ausdehnungslehre*, the next three parts have Grassmann's papers, and

the last part contains a delightful life by Engel; Grassmann's letter to his mother, pp. 16-19, is a literary gem.

Other books than those mentioned above, dealing with the subject, are:

V. SCHLEGEL. *System der Raumlehre.* 1872, 1875.

E. W. HYDE. *Directional Calculus.* 1890.

G. PEANO. *Grundzuge des geometrischen Calculs.* 1891.

The German translation of the work mentioned in the Introduction.

F. KRAFT. *Abriss des geometrischen Kalküls.* 1893.

O. HENRICI and G. C. TURNER. *Vectors and Rotors.* 1903.

Although *all* the ideas of the book which cannot be traced to Hamilton are due to Grassmann, his name does not occur.

E. JAHNKE. *Vektorenrechnung.* 1905.

H. FEHR. *Application de la méthode vectorielle de Grassmann à la géométrie infinitésimale.* 1907.

R. MEHMKE. *Punkt- und Vektorenrechnung.* I. i. 1913.

R. LEVEUGLE. *Calcul Géométrique.* 1920.

For a concise survey of the whole field to which Grassmann's work belongs, and for an extensive bibliography, the reference is naturally to the third volume of the ENCYCLOPÄDIE, where in the second half of the first part there is a double section: *Systeme geometrischer Analyse; I.* by H. ROTHE, pp. 1277-1423, *II.* by A. LOTZE and C. BETSCH, pp. 1425-1595.

An application in a surprising direction is described in A. R. FORSYTH'S *Theory of Differential Equations,* Vol. I., Ch. 5.

R. W. GENESE.

GLEANINGS FAR AND NEAR.

442. Goethe complained that it took him eighteen years of labour to overturn Newton's theory of colours, and although he received the applause of poets and metaphysicians, artists and architects, he could not get a smile of approval from a single physicist.

443. In 1687 the *Principia* of Newton appeared. Sir William (Petty) was one of the few who at once perceived the transcendent merits and importance of the book. "Poor Mr. Newton," he wrote on July 9 of that year to Southwell, "I have not met with one person that put an extraordinary value on his book.... I would give 500*l.* to have been the author of it; and 200*l.* that Charles [his son] understood it."—Lord E. Fitzmaurice, *Life of Petty,* p. 306, 1895.

444. I am perfectly surprised to find a fundamental method of Newton, followed up by Lagrange, unknown to every one I ever came in contact with, whether in books, speech, or writing.—De Morgan to Sir W. R. Hamilton, April 20, 1855.

445. When I was an undergraduate, it happened to me to get very jolly in company with a party who were celebrating the new scholarship of our host. Being, as aforesaid, merry, we proceeded to sing; when it struck one of our party that we could sing as well as the choristers.... We all got our surplices, and stood round the table.... Some one proposed

$$PV.VG : QV^2 :: CP^2 : DC^2 \ldots$$

We tried to make it fit all manner of tunes: I remember "Zitti Zitti," "The Evening Hymn," and "The Campbells are coming." But we left off with a notion that Newton was not so easily set to music.—De Morgan to Sir W. R. Hamilton. Graves, *Life* (iii. 546).

MATHEMATICAL NOTES.

871. [K^1. 7. e.] *Distance-Relations in an Involution Range.*

Reading Mr. Smart's note in the March *Gazette* (p. 333 of this volume), I do not appreciate the point he intends to make in speaking of the four relations under consideration as verifiable separately. There can be no proof or verification of one of them which does not automatically establish the other three; this is, in fact, Russell's argument with regard to them. Let us denote a proposition $\phi(A_1, A_2;\ B_1, B_2;\ C_1, C_2)$ involving three pairs of points by ϕ_{111}, and denote the effect of interchanging the letters in one of the pairs by replacing the corresponding suffix 1 by 2: thus ϕ_{221} stands for $\phi(A_2, A_1;\ B_2, B_1;\ C_1, C_2)$. Then it is the very essence of involution that if ϕ_{111} is any proposition that is a necessary and sufficient condition for involution, each of the eight propositions of the form ϕ_{pqr} is equivalent to ϕ_{111}. What remains to be asked in any particular case is how many of the eight propositions are geometrically distinguishable; the number depends on the form of the proposition, but it is necessarily a factor of eight.

In the case given, we may take for ϕ_{111} the proposition

$$A_1B_2 \,.\, B_1C_2 \,.\, C_1A_2 = -A_2B_1 \,.\, B_2C_1 \,.\, C_2A_1.$$

If this is a necessary condition for involution, it must be sufficient, since it determines one of the points when the other five are given. With this meaning of ϕ_{111}, ϕ_{211} is the proposition

$$A_2B_2 \,.\, B_1C_2 \,.\, C_1A_1 = -A_1B_1 \,.\, B_2C_1 \,.\, C_2A_2,$$

which is geometrically distinguishable from ϕ_{111} but cannot but be an equivalent proposition; the four propositions ϕ_{111}, ϕ_{211}, ϕ_{121}, ϕ_{112} are all distinguishable, but ϕ_{222} is the same proposition as ϕ_{111}, and therefore ϕ_{122}, ϕ_{212}, ϕ_{221} are the same propositions as ϕ_{211}, ϕ_{121}, ϕ_{112}.

If we take for ϕ_{111} the proposition

$$\{A_1A_2,\ B_1C_1\} = \{A_2A_1,\ B_2C_2\},$$

then the eight propositions are indistinguishable when the cross ratios are expressed explicitly, but they are distinguishable from

$$\{B_1B_2,\ A_1C_1\} = \{B_2B_1,\ A_2C_2\}$$

and from

$$\{C_1C_2,\ A_1B_1\} = \{C_2C_1,\ A_2B_2\}.$$

Needless to say, these considerations do not detract from the interest of the point of view which Mr. Smart has discovered. I must protest, however, that the distance-relations emerge, not from the well-known theorem which he quotes, but from a converse to the effect that any set of three pairs of collinear points in involution can be obtained by projection from the pairs of vertices of some quadrilateral.

Perhaps it is worth remarking that a distance-relation between points on a line is necessarily projective if it is homogeneous in each of the *points* involved. The converse is not true, since a projective relation may be modified in form by means of the identity $AB = AC + CB$, but we can infer that one simple projective relation between six points on a line expresses that one product of three distances is a numerical multiple of another product of three distances. If we take the first product as $AB \,.\, CD \,.\, EF$, there is no loss of generality in supposing that in the second product it is C that is associated with A and E with B: the combination $AC \,.\, BD \,.\, EF$ leaves EF as a common factor, and any other combination can be changed into $AC \,.\, BE \,.\, DF$ by an interchange of letters which merely reverses one or more of the factors in the product $AB \,.\, CD \,.\, EF$. Thus the typical relation has the form

$$AB \,.\, CD \,.\, EF = \mu \,.\, AC \,.\, BE \,.\, DF.$$

If we consider this as a homography between C and E, determined by the fixed points A, B, D, F and the multiplier μ, we see that to $C=A$ corresponds $E=F$ and that to $C=D$ corresponds $E=B$. One other correspondence determines the homography, and the homography will be the involution in which A, F and B, D are corresponding pairs if we add the condition that to $C=F$ corresponds $E=A$; thus we have μ determined by the equation

$$AB \,.\, FD \,.\, AF = \mu \,.\, AF \,.\, BA \,.\, DF.$$

That is, $\mu=1$, and we can verify at once that with this value of μ, to $C=B$ corresponds $E=D$. The points being now paired in the involution, we write A', B', C' for F, D, E, and we have the relation in the form

$$AB \,.\, CB' \,.\, C'A' = AC \,.\, BC' \,.\, B'A',$$

that is, $AB \,.\, B'C \,.\, C'A' = -A'B' \,.\, BC' \,.\, CA$, which is one of the four standard forms.

E. H. Neville.

872. [A. 2. a.] *A Problem in Algebra.*

The following reasoning occurs in Chrystal's *Algebra*, vol. i. p. 361:

Let
$$u_1 \equiv a_1x + b_1y + c_1z + d_1 = 0,$$
$$u_2 \equiv a_2x + b_2y + c_2z + d_2 = 0,$$
$$u_3 \equiv a_3x + b_3y + c_3z + d_3 = 0.$$

Let $\Delta = (a_1b_2c_3)$, $\Delta_1 = (d_1b_2c_3)$, $\Delta_2 = (d_1c_2a_3)$, $\Delta_3 = (d_1a_2b_3)$, and let $A_1, B_1, \ldots$ be the co-factors of $a_1, b_1, \ldots$ in Δ.

If $\Delta \neq 0$, then A_1, A_2, A_3 are not all zero. Suppose that $A_1 \neq 0$; then the system is equivalent to

$$u_2=0, \quad u_3=0, \quad A_1u_1 + A_2u_2 + A_3u_3 = 0,$$

that is, to
$$u_2=0, \quad u_3=0, \quad \Delta x + \Delta_1 = 0.$$

Similarly we have $\Delta y + \Delta_2 = 0$ and $\Delta z + \Delta_3 = 0$.

Hence the equations have the unique solution $(-\Delta_1/\Delta, -\Delta_2/\Delta, -\Delta_3/\Delta)$.

This reasoning is wrong. The problem is to say why, and to complete the argument.

In other words, the equations $\Delta x + \Delta_1 = 0$, etc., are to be obtained by a series of steps which are reversible provided that $\Delta \neq 0$. This is what Chrystal has not done. The answer can be given very briefly.

S. Barnard.

873. [L¹. 6. a.] *The Circle of Curvature of a Conic.*

The equation to the circle of curvature may be written in the compact form

$$\begin{vmatrix} s/s_1 & x-x_1 & y-y_1 \\ a-b & \xi & -\eta \\ 2h & \eta & \xi \end{vmatrix} = 0,$$

where $s=0$ is the conic, $s_1=0$ the tangent, $\xi \equiv ax_1 + hy_1 + g$ and $\eta \equiv hx_1 + by_1 + f$. This form of the result follows, of course, from $s = s_1\{p(x-x_1) + q(y-y_1)\}$ by means of the conditions $a-b = p\xi - q\eta$ and $2h = q\xi + p\eta$.

A. Robson.

ROBERT FREDERICK DAVIS

December 11, 1852—May 25, 1927

Honorary Auditor of the Mathematical Association, 1897-1916

REVIEWS.

The Elements of Aerofoil and Airscrew Theory. By H. Glauert. Pp. 228. 14s. 1926. (Cambridge University Press.)

Eighteen years ago, when Mr. Asquith appointed his Advisory Committee for Aeronautics, nothing seemed more certain than that aerodynamics must develop as a purely empirical science; for where its help was needed most,—that is, in explaining the forces encountered by solid bodies in motion through air or water,—Theoretical Hydrodynamics, it appeared, had no remarks to offer. For fluids devoid of friction it could supply a theory of great beauty and of wide range. But that theory, unfortunately, infers that the resistance will be *nil*:—"On this principle", Rayleigh remarked in 1914, "the screw of a submerged boat will be useless, but, on the other hand, its services will not be needed. It is little wonder that practical men should declare that theoretical hydrodynamics has nothing at all to do with real fluids." The crux of the matter may be stated in his words:—"In the extreme cases, when viscosity can be neglected and again when it is paramount, we are able to give a pretty good account of what passes. It is in the intermediate region, where both inertia and viscosity are of influence, that the difficulty is greatest."

That difficulty is not yet overcome, but it has been turned; to-day, although we still have no exact understanding of the action of an aeroplane's wing, the work of Prandtl and his school has supplied what for practice is almost as useful,—a theory which can predict. Remembering that the theory is founded on ideas suggested years ago by Lanchester, we must regret that this country has taken but little share in its subsequent development, and that it has not, except within the last seven years, appreciably influenced our aeronautical design and research. No man has done more to spread the new ideas than Mr. Glauert, who has also made important applications of the theory, notably to the problems of the air-screw and of the "autogyro": now he has laid us under a further debt, by providing, within the compass of one short book, a systematic account of the whole subject.

The book reveals a power of concise exposition which is too seldom devoted to applied science. Its type is large and clear, with margins comfortably wide, and within its 226 octavo pages space has been found for 116 diagrams, clearly drawn to a large scale. Yet it may be read without previous knowledge of hydrodynamics, and its seventeen brief chapters survey the whole ground of modern aerofoil theory. Primarily, as its title shows, it has been planned as a text-book for the student or designer, and it is in this aspect that the arrangement must be judged; the references, for example, are clearly not intended to be exhaustive, but they afford valuable guidance for a more extended study of the subject.

One preliminary chapter, describing the experimental observations which a theory has to explain, is followed by four devoted to the "classical" theory of inviscid fluids, and by two which explain, with admirable brevity, the principles of conformal transformation and, in particular, the special transformations of Joukowski. Thus equipped with analytical weapons, the reader is given a very short account, limited to one chapter, of such progress as has been made towards solving the difficult problems of "skin friction" and the origin of the "viscous drag"; with Chapter IX he begins his study of the vortex theory of aerofoils.

This, of course, is Prandtl's great achievement,—the adaptation of Lanchester's flow pattern, with its wing circulation and trailing vortices, to provide a quantitative treatment of three-dimensional motion round an aerofoil of finite span. One assumption made (of an idealized vortex field, necessarily associated with a lifting aerofoil, and subject to the laws of classical hydrodynamics), purely analytical methods lead to predictions of which it is hard to say which is most striking,—their variety, their accuracy under test, or their direct applicability to design and experiment. Chapters XI-XIII treat of the effects of varying span, gap, number of planes and "stagger"; Chapter XIV, of corrections which must be applied to wind-tunnel measurements; Chapters XV and XVI, of the application of vortex theory to the problem of the airscrew.

No better arrangement could have been adopted for a text-book, and undoubtedly, in this country, a text-book was needed most. But some day, I hope, this book will be followed by another, and then, I have no doubt, Mr. Glauert will fill in the somewhat slender outline of his present Chapter VIII. Perhaps he will consider, when that time comes, the possible advantages of a slightly altered arrangement. For really, as it appears to me, we ought to recognize not one "Prandtl theory", but two. The first forms the main subject of the present work; its methods are those of the classical theory, and its assumptions, based on Lanchester's picture of the flow pattern, are justified, ultimately, by its success in prediction. The second, which develops the notion of the "boundary layer", is more fundamental, more difficult, and (probably) less productive of concrete results than the first; its aim is to explain the circulation round a lifting wing in terms of the known equations of a viscous fluid. The two theories are independent in this respect at least, that either could develop although the other were non-existent.

To unite them as successive parts of a single argument is to achieve a continuity in presentation which must greatly assist the student. But their methods are too distinct to permit a really satisfactory blend, and the combined theory, in the nature of the case, is exposed to criticism from many angles, depending on the standpoint of the reader. The mathematician will base his judgment on the rigour of the analytical processes employed to deduce the kinematic consequences of the assumed vortex field. To the physicist, on the other hand, the state of the analytical superstructure will seem unimportant in comparison with doubts which he must feel about the security of its foundations. For the combined theory seeks to bring phenomena, in their very essence dependent on the viscosity of the fluid and its interaction with the solid boundary, within the scope of analysis which he knows is strictly applicable only to vortex motions existent throughout all time in a fluid devoid of all viscosity. He is prepared to accept the theory pragmatically, on the evidence of its success in prediction; but acceptance of a reasoned argument must be based on other grounds, and it cannot, I think, be claimed that the vortex theory has yet been established by reasoning in which every step is logically sound. There are gaps which must be bridged by arguments plausible rather than rigorous: sometimes these prove too little, more often they prove too much, and we find that a conclusion which we have accepted is found, on experiment, to need correction.

These views imply no criticism of the book which is before me now. Mr. Glauert has chosen the best arrangement for his purpose, and he has shirked no difficulty which that arrangement has entailed. His book gives an admirable account of a fascinating theory, and can be recommended as indispensable to every student of modern aerodynamics.

Only in one small detail have I found matter for regret: it is trivial in relation to the book as a whole, and I should not draw attention to it here, had I not heard a similar fallacy many times before, and from men who should know better. I refer to the statement (made not once but several times) that vortex sheets may be conceived as a succession of point vortices "which act in the manner of roller bearings". If it be contended that this concrete illustration will help the beginner towards a comprehension of the difficult and purely mathematical concept of vorticity, I would reply that, so far as it gives him an idea at all, that idea is incorrect: he will misunderstand either the vortex sheet, or the action of a roller bearing. For a vortex sheet is the limiting case of a layer of fluid in laminar shearing motion: there is no feature in the latter motion corresponding to the discrete roller,—and a flow pattern is not altered by reduction of its scale! R. V. SOUTHWELL.

The Structure of the Atom. By E. N. DA C. ANDRADE. Third edition. Pp. xx + 750. 30s. 1927. (Bell.)

A writer whose mind is sufficiently cultivated to be able to assimilate the latest developments of physical science, and who at the same time has sufficient energy to write an expository treatise and bring it as nearly as possible up to date, is performing a great service to his generation, and indeed is assisting the progress of science itself. Such a service has been performed by

Dr. E. N. da C. Andrade, the indefatigable Professor of Physics at the Artillery College, Woolwich, who has spared no pains to make himself well acquainted with the developments in atomic and spectroscopic theory during the last fruitful and eventful years. His treatise was first published in 1923, was received with gratitude and widely read. A second edition, practically unchanged from the first, appeared in 1924; and now it has been found necessary to double its size and practically to rewrite it for a third edition. The book in its original form is well known: it may suffice to indicate the sort of additions that have had to be made. Nor can this edition be considered in any respect final, for it is a subject which is attracting the attention of a multitude of competent workers in every country of the world, and every year witnesses new discoveries and fresh developments.

In particular some recent investigations developing the theory of the spinning electron, to which so much attention has been paid in this country of late by Professor Allen and others, will probably involve, not exactly a recasting, but a most important supplementing, of some parts of the book. But as Professor Andrade says in his Preface, if he waited to bring the book up to date at the present rate of progress in physical science its publication would be much delayed. In a footnote to p. 555 the author does refer to some important work by Uhlenbeck and Goudsmit on this subject, in which a special quantum number is allotted to electron spin; but since then there has been a heroic attempt by Professor Louis V. King, Macdonald Professor of Physics at Montreal (published last autumn), to rescue the whole phenomena from quantum considerations, and replace these by orthodox dynamics—with what success remains to be seen. The attention of mathematicians may also be usefully directed to Professor Whittaker's ingenious and hopeful efforts in this purely dynamical direction.

All workers on the subject will be grateful to Professor Andrade for his admirable and comprehensive survey of what was known, say, two years or even one year ago. It may hereafter be found that in the gyromagnetic electron, or what might be called an electrified magnet, there may be a possibility of accounting for a great many of the quantum and other phenomena, which hitherto have seemed mysterious, by ordinary straightforward dynamical methods; and if such an investigation holds water and withstands criticism it will (philosophically) be an immense step in advance. It might plausibly be urged that with all the latitude inevitably accompanying a full treatment of the behaviour of electrified magnets, of almost infinitesimal inertia and dominated by prodigious forces, it would be odd if a number of phenomena could not be accounted for; and if the present undynamical quantum theory does lead to an exact account of the multiplex structure of spectral lines, as the work of Sommerfeld, Heisenberg, and others, seems to make extremely probable, the procedure will be justified *a posteriori*, even though some of the arguments used are more akin to those of thermodynamics and probability than to the more directly mechanical and individual treatment which ultimately may be hoped for.

Meanwhile and apart from any supplementary theory (attempts at which probably ought not to be taken as substantiated until they have been further examined), Professor Andrade's book gives a satisfactory summary of what is known about line spectra, and about the atomic perturbations that account for the fine and multiplex structure of those lines. The present edition differs considerably from the first edition; it has largely been rewritten and is much supplemented. In particular the treatment of the magnetic properties of atoms has been enlarged, and Compton's results, embodying a combination of wave theory and quantum theory, have necessitated a new additional chapter (chap. xviii). Again there is a wholly new chapter (xv) on multiplex structure and anomalous Zeeman effect, which must have cost the author a great deal of labour, for it is no easy task to assimilate the results of a number of workers at a subject in an intricate and rather confusing interim condition.

But indeed there are many new features in this edition, and among them we may note the Ehrenfest theory of adiabatic invariants. This theory aims at overcoming the difficulty of the unrestricted quantum theory, whereby electron orbits were strictly limited to certain energy levels; so that though the particle could jump from one orbit to another, with either emission or

absorption of radiation, it could not apparently be perturbed continuously into a modified orbit by slow forces, such as could readily be applied artificially, say by a magnetic field. Intermediate orbits were assumed impossible; or at least none but quantised orbits are supposed stable. A change from one orbit to another is brought about by a sudden change of energy (interchange with the ether), either absorption or emission, and this sudden change has to be a whole quantum, and must be proportional to a certain frequency.

But Ehrenfest has pointed out that a change may be made in the energy and the frequency conjointly, so as still to satisfy quantum conditions, if energy is supplied or withdrawn slowly. And this is likened to an adiabatic operation in heat, which, though usually in practice effected suddenly, would be more perfectly and completely done slowly, if only certain practical conditions could be satisfied, such as walls impervious to heat. A mechanical analogy to this adiabatic method of introducing extraneous energy has been suggested, namely, either a simple or a conical pendulum, with a ring on its string, so that it could be lengthened or shortened. Suppose it gradually and slowly shortened:—the frequency increases, but work is done by the movement of the ring, and the energy increases too, so as still to be proportional to the frequency. If however the ring is suddenly jerked down at an instant when the pendulum is vertical, no work is done, and proportionality is violated. The orbit, so to speak, is then a different one.

This analogy seems to have been suggested by Lorentz as a conference question or objection, and was at once taken up, answered, and adopted, by Einstein. The energy and the frequency may change, and will change together, so long as the moment of momentum does not change.

Among other new features in this edition may be noted:—The description of Saha's work (with a diagram, worked out by the author for this book, enabling the percentage ionisation for a given temperature, pressure, and ionisation-potential to be read off—page 351); Einstein's proof of Planck's formula—page 353 *et seq.*; the treatment of the Stark effect by the correspondence principle; Bohr's derivation of the Rydberg-Ritz formula; and the possibility of the simultaneous displacement of two electrons—page 562.

In a concluding chapter the author tries to sum up his own present position on various controversial and doubtful points, much as he did in the first edition, but of course now with differences. He commends the work of Heisenberg in dispensing with ideas that cannot be brought to the test of experiment, and in employing more general if vaguer methods (somewhat as Einstein preferred to work without explicit reference to the ether), using only those phenomena that are susceptible to observation, while at the same time not controverting the reality and ultimate necessity of ignored mechanism: ignored only because of our present ignorance as to the details of its working.

This is an imperfect indication of some of the many points that appear in the new edition. To refer to the whole of a book like this with any brevity is impossible: indeed a review would be long delayed if the whole of it had to be read. There is obvious food for thought in all recent developments in atomic theory, the success of which, as tested by spectroscopy, though as yet incomplete and in complex atoms doubtful, is still distinctly marvellous.

Oliver Lodge.

Practical Geometry. By A. W. Siddons and R. T. Hughes. Pp. x + 264. 4s. 1926. (Cambridge University Press.)

This book, like the authors' *Theoretical Geometry*, reviewed in the March number of the *Gazette*, to which it is intended to be a companion volume, is based on the various geometry books of Godfrey and Siddons. In most cases a blend of the two books is required, but the blendings differ according to the ages, requirements, opportunities and abilities of the pupils. In this book appeal is made to intuition as well as to experiment and easy deductive reasoning. The topics are so clearly and cleverly expounded that a reader's appreciation of *theoretical geometry* does not suffer, but rather the reverse. Throughout the book three-dimensional exercises are introduced, and there are special chapters devoted to solid geometry. The exercises, chiefly numerical and practical, are numerous, well chosen, and carefully graduated.

W. J. Dobbs.

Vorlesungen über höhere Geometrie. By Felix Klein. 3rd edition. Edited by W. Blaschke. (Grundlehren der Mathematischen Wissenschaften, Bd. xxii.) Pp. vii + 405. 24 marks. 1926. (Springer.)

Even those who possess the lithographed edition (1893) of Klein's lectures on higher geometry will be glad to have this legible edition in the well-known valuable series of yellow books published by Springer. The editor is W. Blaschke; he has reprinted volume one without essential modification, merely adding references to papers and books which have appeared since 1893 and modernising the headings of the paragraphs. We may note also as new the attribution of the "pentacycle" to the Greek Stephanos, who died in 1917 and whose work is not so well known in England as it should be. (I have in my possession an appreciation of his work by his colleague at Athens, Prof. Remoundos, but my knowledge of modern Greek is not sufficient for me to make it out with any ease.)

The second volume of the old edition, which dealt with continuous and discontinuous groups, has not been reprinted; Blaschke remarks that it has only a very slight connection with the first volume and that in any case it would require to be completely rewritten. In place of this the editor has added a section giving an account of some recent lines of investigation in geometry. This part of the book naturally attracts the attention of one who is familiar with the old edition. It is interesting and valuable, but it must be confessed that it is much harder reading than the rest, partly, no doubt, because the subjects dealt with are less familiar and more difficult, but also because Prof. Blaschke, with all his virtues, has not got quite that gift of lucidity which makes Klein's books so wonderful. It could hardly have been expected.

The first article in the additions deals with Study's work in line-geometry. Study showed that the four dimensional line-geometry of ordinary space comes to the same thing as geometry on the surface of a sphere if we introduce a special kind of complex point. The complex numbers involved are the "dual" numbers $a+\epsilon b$ of Clifford, in which a,b are real and the new unit ϵ satisfies the rule of calculation $\epsilon^2=0$. If, referred to rectangular axes, the Plücker coordinates of a straight line are l, m, n, l', m', n' (the first three being direction-cosines) and we put $L=l+\epsilon l'$, etc., we have

$$L^2+M^2+N^2=l^2+m^2+n^2+2\epsilon(ll'+mm'+nn')=1,$$

on account of the relations $l^2+m^2+n^2=1$, $ll'+mm'+nn'=0$. The dual points of the unit sphere thus give an image of the real, directed straight lines of Euclidean space. It easily follows that the spherical distance of two dual points is equal to $\phi+\epsilon\psi$, where ϕ is the angle between the corresponding straight lines and ψ is their shortest distance. One curious consequence is that Morley's remarkable theorem on the common perpendiculars obtained from three skew lines turns out to be the same as the simple result that a spherical triangle has an orthocentre.

Study's method was developed in his book *Geometrie der Dynamen* (Leipzig, 1903); and Blaschke himself has applied it to the differential geometry of ruled surfaces in his text-book *Differentialgeometrie*, in this same series. Further applications and extensions are considered here, and the article, which should be read in connection with four paragraphs of Fano's article in the *Encyklopädie* (Bd. iii. 1, 1, p. 325), is all the more valuable since Study is so difficult to read.

Another article is on Analysis Situs, that extraordinarily difficult subject which has made so much progress recently, especially in Holland under Brouwer, and which has begun to attract the attention of English mathematicians (see an expository article in the *Math. Gazette*, December, 1926, p. 222). Blaschke confines himself to giving examples of the kind of problem which has been attacked in each of the two branches of the subject; in that connected with *Mengenlehre* he gives a sketch of Alexander's proof of Tietze's deformation theorem—note his expression "Kautschukterminologie," which is worthy of Klein at his best—and then shows that things are not always what they seem; in the *Combinatory* theory he deals with the theory of knots and plaits (if that is the right way to translate "Zöpfe").

Other topics treated are Levi-Civita's *parallel-displacement*, the differential equation of Monge and its relation to the calculus of variations and the theory of contact-transformations, and the theory of elementary divisors (invariant factors). It is remarkable that this last matter, which has so many applications in geometry, differential equations and dynamics, should not yet have been developed in a straightforward, simple way. Kronecker's papers make impossible reading, Muth's text-book is worse, and Bromwich's tract, *Quadratic Forms*, while the best introduction to the subject, is not altogether satisfactory. Blaschke here gives a sketch of a method, due to Weyl, which uses the theory of matrices. But surely even this is far too complicated, and he is careful to exclude the "singular case," which has received such a beautiful and simple geometrical treatment at the hands of Segre. F. P. W.

Mathematical and Physical Papers, 1903-1913. By Benjamin Osgood Peirce. 1926. (Harvard University Press.)

Professor Peirce is probably best known on this side of the Atlantic as the author of the only text-book in the English language which deals definitely with the "Theory of the Newtonian Potential," a book which, in the revised and enlarged form of its third edition, is a necessary work in any mathematical library. The collection of his mathematical and physical papers just published is a reminder that Professor Peirce, during his tenure of the chair of Mathematics and Natural Philosophy at Harvard University from 1884 to 1914, was thoroughly acquainted with the practical applications of the potential theory.

The physical papers deal largely with problems connected with the various magnetic properties of iron and steels. The series appears to be the outcome of investigations pursued in the Jefferson Physical Laboratories with a view to improving the constancy of the permanent magnets used in moving coil galvanometers. The subsequent researches deal with many of the problems arising out of the original investigation.

Many of the papers in the series have historical interest only, partly due to the progress in technique since the original papers were published, and partly owing to the inherent difficulties of standardising magnetic data on iron and steels. On the other hand, many deal with important branches of magnetism, which the modern student, reading more quantum theory than he can digest, is apt to disparage as being "classical." Amongst the latter may be mentioned the author's proofs, from both the theoretical and the experimental side, that the so-called "von Waltenhofen effect"—the apparent dependence of the magnetic induction produced in an iron rod on the rate of growth or decay of the magnetising field—was due solely to eddy currents in the iron.

Other papers contain useful information for the experimenter which is not usually easily accessible. Included in these is a thorough investigation of the necessary qualities of a good fluxmeter, and also a discussion of the errors of the anchor-ring method of determining B-H curves.

The substance of the few mathematical papers has been incorporated in the last edition of the book referred to above, and does not call for any comment. The publishers themselves point out that most of Professor Peirce's work is "buried in the proceedings of such learned societies as the American Academy of Arts and Sciences." This collection of papers will therefore be all the more useful to all those interested in magnetic phenomena. W. Sucksmith.

Traité du Calcul des Probabilités et de ses Applications. Tome IV, Fascicule I. **Applications au Tir.** By J. Haag. Pp. vi+184. 25 frs. +40% majoration. 1926. (Gauthier-Villars.)

Gunnery is a classical example of the theory of probability, and M. Borel's admirable treatise would have been incomplete without some account of this fascinating example. He is singularly fortunate in having secured the services of M. Haag, who was engaged, during the war, with the Commission de Gâvre and subsequently at the French Centre of Artillery Studies, and is therefore able to combine with his profound mathematical knowledge a considerable

amount of practical experience. This fascicule, which forms the first part of Tome IV of the treatise, is the outcome of the author's work during and since the war. It contains a number of original investigations, among which may be mentioned the calculation of mean range and probable error, ranging a gun or battery, the probability of fire for effect and barrage fire, and the experimental verification of Gauss' law.

The classical methods of calculating the chances of hitting targets of various shapes are treated in detail. There follows an interesting discussion on the calculation of the mean range, and the author gives his opinion, with which we entirely agree, that the criterion for rejection of abnormal rounds should be independent of the number of rounds fired in the series.

The determination of the probable error or "fifty-per-cent" zone is discussed; fuse burning is briefly treated and the difficulty of interpolating probable errors from limited firing results is successfully surmounted. The errors to be expected when engaging a target and the method of dividing a circular target into rings of equal probability are next detailed.

These considerations lead us to the problem of ranging. The French artillery methods are reviewed and criticised from the theoretical point of view. It is shown that it is generally better to stop ranging when a contradiction or hit is obtained, rather than to verify by firing the short bracket.

Firing for effect is considered from the points of view of efficiency and accuracy, and a short chapter is devoted to game shooting.

The book concludes with interesting notes on the experimental verification of Gauss' law—the agreement between theory and experiment is shown to be such that the normal law may safely be taken as a basis for rules of fire—choice of mean range, calibration of the guns of a battery and barrage fire.

The book is a valuable addition to M. Borel's treatise, and will be welcomed by all who desire an elementary account of the subject but do not of necessity possess more than a knowledge of the general principles of the calculus of probability. F. R. W. Hunt.

Fourfold Geometry. By D. B. Mair. Pp. viii + 184. 8s. 6d. 1926. (Methuen.)

This book is of a type which puzzles me. I cannot believe for a moment that by any study of algebra we develop a spatial intuition in four dimensions, and the sentence (p. 18), "To the reference frame . . . carried by O we add a fourth axis, which we look on as perpendicular to the original three axes," is to me absolute nonsense. Admitting that there does seem to be validity of some kind in arguments which depend only on parallel projection, and that a sentence like "As OP falls away in any direction from the t-axis in any direction, it reaches a position where $x^2+y^2+z^2=t^2$" does seem to mean something, the case is given away when a locus is said to resemble an hour-glass "as nearly as a three-dimensional locus can resemble a two-dimensional one". We may feel that in some way a circle resembles a sphere, but we have only to ask what curve resembles a hyperboloid of one sheet to realise that imagination is in fact impotent to climb a dimension. If this is true when the fundamental quadratic form is definite, matters are worse when the form is indefinite, and since Mr. Mair's object is to help students of relativity, it is the indefinite form whose simplest expression is $t^2-x^2-y^2-z^2$ with which he is dealing; hence we have such warnings as that "In our picture the size shown for OU is a quasi-Euclidian size $\sqrt{(t^2+z^2)}$, whereas the true size is the much smaller quantity $\sqrt{(t^2-z^2)}$."

Mr. Mair's account of the algebra of the eventful fourfold is clear. He teaches his readers to appreciate the flexibility which accompanies a readiness to use an arbitrary frame of reference and the importance of invariants when axes are freely changed, and to acquire a familiarity with the elementary conventions of the tensor calculus that will rob the early chapters of more serious treatises of most of their terrors. Is it worth while to do so much for them while assuming not merely that they would not understand an intelligent discussion of the question "What is the basis of the whole investigation?" but that they will not see through the most transparent bluff? E. H. N.

Compléments de Géométrie Moderne. By C. Michel. Pp. 317. 25 frs. +40% majoration. 1926. (Vuibert, Paris.)

This is a series of chapters on systems of conics, cubic curves, quadrics, Steiner quartic and ruled cubic surfaces, based on cross-ratio and parametric representation. The choice of topics appears governed by the wish to use as few and as special methods as possible.

There is no index and no detailed table of contents; no preamble of any kind, no digressions, and no attempt to relate the subject-matter to larger geometrical notions. The book is probably meant for students with little time, no access to literature, and previous knowledge confined to the author's lectures of the year before. The only way to get them a suitable textbook is no doubt to write it, and to limit the range severely; but it does not make the book generally useful, nor does it justify such amazing liberties as the definitions given of quadratic transformation and involution, neither of which covers inversion.

In spite of all this, many familiar metrical properties are deduced with surprising neatness; and any lecturer could find something in the text or examples to incorporate in his own course. H. P. H.

Plane Analytical Geometry. By I. A. Barnett. Pp. vi+269. 10s. net. 1926. (Chapman & Hall.)

This book aims at presenting the material "informally and naturally, without the use of special devices, even at the expense of lengthening the proofs". This is a worthy aim, and on the whole the author is successful in reaching his mark. He makes a very special point of the "algebra-geometry dictionary" which is used throughout and "is intended to make the student conscious of the analytic method". The dictionary is compiled as required in the course of the text, and is repeated in full at the end of the book. Most teachers of analytical geometry have such dictionaries as this in their minds and may even cause students to construct them; but in the present case it is possible that a little too much has been done for the learner.

The book contains eight chapters, of which the first five consist of straightforward explanations of the line, circle, and conic sections, with excellent illustrative examples and plenty of good exercises. The last three chapters are refreshingly written, the headings being: Chapter VI, Some general methods for the solution of locus problems; Chapter VII, Some applications to curve tracing; Chapter VIII, Polar coordinates. Here the reviewer makes a confession of faith: he believes in the might of parameters, the virtue of polar (and other) graph paper, the wisdom of including in such a book as this an account of what Edwards calls "Some well-known curves", and the beauty of the method of abridged notation.

It appears to be contrary to the author's aim of a "natural" development and the avoidance of special devices, to include an account of the method of abridged notation which is surely the whole point of Descartes' wonderful invention. But it is most unfortunate that on p. 53, where the method is hinted at, the author (rather plaintively) writes: "We must remember however that this method does not give the coordinates of the point of intersection." Such a remark is definitely misleading. The essence of the method is that we do not have to find the "coordinates of the point of intersection".

The book is written in clear and vigorous language, but it is not at all obvious why points of inflexion should be singled out for re-christening, and be called "bend-points". Why should not tangents be called "touch-lines"? The word tangent is used as an adjective, "circles are tangent internally". Perhaps these are pet phrases of the author and not part of a settled scientific jargon.

But when all has been said by way of adverse criticism, this remains a good book; suitable as a first and final course for pass students and as an introduction for honours students. Furthermore, it is exceedingly well printed and bound, and there is a full index. The author and publishers are alike to be congratulated. N. M. Gibbins.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows :

From Mr. **F. J. Cock,** the first volume of a work of which he gave us the second volume last year :

J. Bauschinger and J. Peters		
	Logarithmic-Trigonometrical Tables with eight decimal places ; I : Logarithms of all Numbers from 1 to 200,000 - - - - - - -	1910

From the Author :

R. W. M. Gibbs	Algebra to the Quadratic - - - - - -	1927

From the Library of the late Sir **George Greenhill,** by the generosity of Miss Greenhill and Mr. H. H. Greenhill :

O. S. Adams	Polyconic Projections - - - - - - -	1919
A. B. Basset	Hydrodynamics ; I - - - - - - -	1888
	Can any member add the second volume ?	
H. Bell	Solution of Spherical Triangles - Edinburgh 5	1915
C. Bourlet	Statique Graphique - - - - - - -	1910
K. Brandenberger		
	Didaktik des mathematisch-naturwissenschaftlichen Unterrichts - - - - - - -	1920
W. E. Byerly	Generalized Coördinates in Mechanics and Physics -	(1916)
	Integral Calculus (2) - - - - - - -	1889
P. Carus	Foundations of Mathematics - - - - - -	1908
J. W. Colenso	Plane Trigonometry ; I ( ) - - - - - -	1863
	The work was completed by a second volume.	
A. W. Conway	Relativity - - - - - - Edinburgh 3	1915
H. T. Davidge	Notes on Optics - - - - - - - -	1908
N. M. Ferrers	Spherical Harmonics - - - - - - -	1877
L. R. Ford	Automorphic Functions - - - - Edinburgh 6	1915
R. Fricke	Hauptsätze der Differential- und Integral-Rechnung (2 vols.) - - - - - - -	1897
H. B. Goodwin	Plane and Spherical Trigonometry (2) - - -	1888
A. Gray and G. B. Mathews		
	Bessel Functions - - - - - - - -	1895
O. v. Guericke	Neue "Magdeburgische" Versuche 1672 ; Ostwald 59	1894
	Translated from Latin by F. Dannemann.	
T. G. Hall	Calculus (3) - - - - - - - - - -	1841
H. Hancock	Maxima and Minima - - - - - - -	(1917)
H. Heilermann	Bestimmung von Grösten und Kleinsten Werthen -	1871
E. L. Ince	Descriptive Geometry and Photogrammetry Edinburgh 1	1915
J. H. Jeans	Problems of Cosmogony and Stellar Dynamics -	1919
W. W. Johnson	Differential Calculus - - - - - - -	1904

C. F. KLEIN	On Riemann's Theory of Algebraic Functions and their Integrals - - - - - - - - -	1893
	Translated from German by F. Hardcastle.	
	Über eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen -	1904
C. G. KNOTT	Life and Scientific Work of P. G. Tait - - -	1911
J. H. LAMBERT	Entwerfung der Land- und Himmelscharten 1772; Ostwald 54	1894
J. LARMOR	Aether and Matter - - - - - - -	1900
P. M. H. LAURENT	Calcul des Probabilités - - - - - - -	1873
W. J. LOUDON	Rigid Dynamics - - - - - - - -	1896
J. H. PRATT	Attractions, Laplace's Functions, and the Figure of the Earth {3} - - - - - - - -	1865
N. QUINT	De Wervelbeweging - - - - - - -	1888
G. ROSENHAIN	Functionen zweier Variabler mit vier Perioden 1851; Ostwald 65	1895
	Translated from French by A. Witting.	
H. SCHUBERT	Unterrichts- und Vorlesungspraxis (3 vols.)	1905, 1905, 1906
M. STEGEMANN und L. KIEPERT	Differential- und Integral-Rechnung (2 vols.) {7, 5}	1895, 1894
H. E. TIMERDING	Die Fallgesetze - - - - - - - -	1912
F. F. TISSERAND	Exercices sur le Calcul Infinitésimal {2} - - -	1896
	With a section by P. Painlevé.	
F. F. TISSERAND et H. ANDOYER	Cosmographie - - - - - - - -	1895
W. A. WHITWORTH	Choice and Chance {2} - - - - - - -	1870
A. WITTING	Einführung in die Infinitesimalrechnung - -	1912

Also pamphlets by T. J. I'A. Bromwich, R. K. Hezlet, and D. E. Smith, several volumes of periodicals, including a long but imperfect run of *L'Enseignement Mathématique*, the separate issue of the

Enquête sur la Méthode de Travail des Mathématiciens - - 1908

edited for that journal by H. Fehr, the volume of monographs edited by M. Merriman and R. S. Woodward and published as

Higher Mathematics - - - - - - - - - - - - 1896

and the following publications of the C.I.E.M., of which Sir George Greenhill was a Vice-President:

Comité Central - - - - - - - - - - - 8

Sous-Commissions Nationales:

Allemagne	16, 24, 46, 47, 48, 57, 59, 60, 62	Danemark	90-99
		Russie	267
Autriche	84	Suisse	294

From Mr. **W. J. Greenstreet**:

T. ALEXANDER and A. W. THOMSON	Scientific Design of Masonry Arches - - -	1927
J. KEIL	Euclid's *Elements* {4; S. Cunn} - - - - -	1741
	A volume with some movable diagrams for solid figures.	

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY. MATHEMATICAL NOTES.

Several volumes of the *Proceedings* and several numbers of the *Notes* came to the Association from Sir George Greenhill's library, and in exchange for numbers of the *Gazette* missing from their collection the Edinburgh Mathematical Society has filled all the gaps that remained here. We have therefore complete sets, that is, the *Proceedings* from 1883 (44 vols.) and the *Notes* from 1909 (23 nos.).

The current volume of the *Proceedings* begins a new series. A larger page simplifies the printers' problems, and the first number, which was issued in May, sets a high standard in both interest and appearance. We wish the journal in its new form every success.

Except for official records and a six-page note by C. G. Knott, the volume of the *Proceedings* for the Session 1883 consists of a treatise by J. S. Mackay, then President, on the geometry of the triangle; the volume is numbered 1 in the series, but the date of publication was 1894.

SUMMER MEETING OF THE YORKSHIRE BRANCH.

The summer meeting of the Yorkshire branch of the Mathematical Association was held on Saturday, May 21st, at the Staff House, Leeds University. Twenty-two new members were elected. The Chairman, Professor S. Brodetsky, gave an account of the Newton celebrations at Grantham, and Mr. A. E. Ingham, M.A., Reader in Mathematical Analysis at the University of Leeds, gave an instructive paper on "The Map Colouring Problem". Mr. J. H. Everett, B.Sc., Principal of the Leeds Technical College, opened a discussion on Mr. C. Cooper's paper on "The Introduction of Algebra into the School Course".

WANTED.

GAZETTE No. 44. Five shillings is offered for a copy, and any member with one to spare is asked to send it to the Librarian.

446. Newton, meeting Bentley in town, asked him how philosophical pursuits were going on at Cambridge. "There are none," was the reply, "for you kill all the game and leave us none to pursue." "Not so," retorted Sir Isaac, "you may start game in every bush if you will but beat for it."

447. In *Sphinx-Oedipus* some years ago the followingwas suggested as a new symbol:

$$a+b+c\overset{3}{=}x+y+z+t$$

to represent the set of three equations

$$a^n+b^n+c^n=x^n+y^n+z^n+t^n, \quad n=1, 2, 3;$$

or again,

$$a+b+\ldots\overset{\substack{6\\4\\2}}{=}x+y+\ldots$$

for the set of three equations

$$a^2+\ldots=x^2+\ldots; \quad a^4+\ldots=x^4+\ldots; \quad x^6+\ldots=x^6+\ldots.$$

448. A regular polygon containing 17, 257, 65537, etc. sides, is capable of being inscribed . . . and, in general, when the number of sides may be denoted by $2n+1$, and is at the same time a prime number.—Leslie, *Elements of Geometry*, 1809, p. 461, n.

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THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"*I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto.*"—BACON (Preface, *Maxims of Law*).

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.